Exam 3

Math 253, Spring 2021

Name: _____

Tell me what you know: If you don't know the entire answer, showing a formula or writing something illustrating that you understand any concept involved in the problem will allow me to give partial credit. I have to give you a 0 if you write nothing down. You will earn more points by writing down one true thing and no nonsense things than you will by writing down one true thing and a bunch of nonsense things.

Show your work: If you use a result from the textbook that we haven't learned in class, tell me where you found it in the textbook. If you do partial fraction decomposition, show how you solved for your constants. This exam is not testing to see if you can get the right answers. It's testing to see if you know calculus.

Check your answers: Take the time before you turn in your test to make sure you have read the directions correctly and in their entirety, that all your work shown is correct, and that you have clearly stated your answer (by boxing or circling it where appropriate).

Pace yourself: If you're stuck on a problem, move on and come back to it later. Don't risk forcing yourself to give partial answers if you run out of time near the end of the test. Do the easy ones first. There are 94 points on this exam. That means you should budget about 0.5 minute(s) for each point a problem is worth in order to complete the exam in time.

Reminder: The only internet resource that you are allowed to use in conjunction with the exam are the notes that I have written and posted to Canvas. If you use any other online resources or if you communicate with other individuals during the exam, you will receive a zero on the exam and you will be reported to the university for academic misconduct.

1. **Vocabulary** (5 points each)

   (a) $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$ is the Taylor series centered at $a = 0$ for some function $f(x)$.

   i. What is the **third Taylor polynomial** for $f(x)$?

   $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10}$

   ii. What is the **eighth Taylor coefficient** for $f(x)$?

   $\frac{1}{65}$

   (b) What is the **interval of convergence** for a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$? Note: I'm looking for the definition here, not a description of how to find the interval of convergence.

   The interval of convergence is the set of all x-values for which $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges

2. **True/False** (5 points each). For each statement, *circle* the word "true" if the statement is true and "false" otherwise. If you're writing on your own paper, write down the *entire* word "True" or "False." No work is necessary, but you can show work for some partial credit if you get the answer wrong.

   (a) **True** or **False**? It is possible for a power series to converge for $x \in (-3,-1) \cup (1,5)$ but diverge at $x = 0$.

   (b) **True** or **False**? If $f(x)$ is a power series which has radius of convergence equal to 3, then any antiderivative of $f(x)$ is a power series with radius of convergence equal to 3.

   (c) **True** or **False**? If $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ both have interval of convergence equal to $(-1,1)$, then $\sum_{n=0}^{\infty} (c_n + d_n) x^n$ also has to have interval of convergence equal to $(-1,1)$.

3. (10 points) Give an example of a power series with center equal to -1 and radius of convergence equal to 6. You don't have to show any work to compute anything, just give the example.

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{6^n}$$

4. (10 points) Give an example of a power series with interval of convergence equal to $[-1, 1)$. You don't have to show any work to compute anything, just give the example.

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

5. Let $h(x) = \arctan((3x)^2)$. $= \arctan(9x^2)$

(a) (16 points) Find a power series centered at $a = 0$ which converges to $h(x)$ by relating $h'(x)$ to a geometric series.

Recall: $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

$$h'(x) = \frac{18x}{1+(9x^2)^2} = 18x \cdot \frac{1}{1-(-(3x)^4)}$$

$$= 18x \cdot \sum_{n=0}^{\infty} (-(3x)^4)^n$$

$$= 18x \cdot \sum_{n=0}^{\infty} (-1)^n (3x)^{4n}$$

$$= \sum_{n=0}^{\infty} 18 \cdot (-1)^n 3^{4n} \cdot x^{4n+1}$$

$$\rightarrow h(x) = \int h'(x)\,dx = C + \sum_{n=0}^{\infty} \frac{18(-1)^n 3^{4n} \cdot x^{4n+2}}{4n+2}$$

$$0 = \arctan(0) = C + 0 = C$$

$$\rightarrow h(x) = \sum_{n=0}^{\infty} \frac{18 \cdot (-1)^n 3^{4n} \cdot x^{4n+2}}{4n+2}$$

For $h(x) = \arctan((2x)^3)$

$$h'(x) = \frac{3(2x)^2 \cdot 2}{1+(2x)^6} = \frac{24x^2}{1-(-(2x)^6)} = 24x^2 \sum_{n=0}^{\infty} (-(2x)^6)^n$$

$$= 24x^2 \sum_{n=0}^{\infty} (-1)^n \cdot 2^{6n} \cdot x^{6n} = \sum_{n=0}^{\infty} 24 \cdot (-1)^n \cdot 2^{6n} \cdot x^{6n+2}$$

$$\rightarrow h(x) = \int h'(x)\,dx = C + \sum_{n=0}^{\infty} \frac{24 \cdot (-1)^n \cdot 2^{6n} x^{6n+3}}{6n+3}$$

$$0 = h(0) = C + 0 \rightarrow C = 0$$

$$\Rightarrow h(x) = \sum_{n=0}^{\infty} \frac{24 \cdot (-1)^n \cdot 2^{6n} \cdot x^{6n+3}}{6n+3}$$

(b) (12 points) Give an interval of x-values on which your power series from part (a) converges. Provide justification for your answer.

The power series for $h(x)$ has the same radius of convergence as the power series for $h'(x)$. The power series for $h'(x)$ converges if and only if $|-(3x)^4| < 1$, i.e. $|3x|^4 < 1$

i.e. $|3x| < 1$, i.e. $|x| < \frac{1}{3}$.

Hence, the power series for $h(x)$ converges on $(-\frac{1}{3}, \frac{1}{3})$

6. (16 points) Let $g(x) = \sin(3x)$. The Maclaurin series for $g(x)$ is $\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{(2k+1)!} \cdot x^{2k+1}$. Use the Maclaurin series and Taylor's Theorem with Remainder to find an upper bound for $\left|R_1\left(\frac{\pi}{2}\right)\right|$.
Suggestion: First, identify a, x, I, and k.

$a = 0$ $\quad k = 1$ $\quad x = \frac{\pi}{2}$

$I = [0, \frac{\pi}{2}]$

Know: If $|f^{(k+1)}(z)| \le M$ for $z \in I$, then

$$|R_k(x)| \le \frac{M}{(k+1)!} |x-a|^{k+1}$$

$\downarrow$ If $|f^{(2)}(z)| \le M$ for $z \in [0, \frac{\pi}{2}]$, then

$$|R_1(\tfrac{\pi}{2})| \le \frac{M}{2!} \cdot |\tfrac{\pi}{2} - 0|^2$$

$f(x) = \sin(3x) \to f'(x) = 3\cos(3x) \to f''(x) = -9\sin(3x)$

Note: $|f''(x)| = |-9\sin(3x)| \le \underset{M}{9}$ for $x \in [0, \frac{\pi}{2}]$

$$\to |R_1(\tfrac{\pi}{2})| \le \frac{9}{2!} \cdot |\tfrac{\pi}{2} - 0|^2 = \frac{9\pi^2}{16}$$